J. DIFFERENTIAL GEOMETRY
16 (1981) 205–212

# SOME AXIOMS OF EINSTEINIAN AND CONFORMALLY FLAT HYPERSURFACES

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## 1. Introduction

The following theorems state main results from the theory of axioms of submanifolds in Riemannian geometry. This theory was initiated by E. Cartan with the *axiom* of *n-planes* [2].

**Theorem A** (*D. S. Leung and K. Nomizu* [10]). *A Riemannian manifold of dimension $m > 2$ is a real space form if and only if it satisfies the axiom of n-spheres, $2 \leqslant n < m$.*

**Theorem B** (*J. A. Schouten* [13]). *A Riemannian manifold of dimension $m > 3$ is conformally flat if and only if it satisifes the axiom of totally umbilical n-dimensional submanifolds, $3 \leqslant n < m$.*

**Theorem C** (*K. L. Stellmacher* [14]). *A 3-dimensional Riemannian manifold is conformally flat if and only if it satisfies the axiom of umbilical surfaces.*

**Theorem D** (*K. Yano and Y. Mutô* [18]). *A Riemannian manifold of dimension $m > 3$ is conformally flat if and only if it satisfies the axiom of totally umbilical surfaces with prescribed mean curvature vector.*

Many axioms of submanifolds were considered in Kaehlerian, Sasakian, pseudo-Riemannian and other manifolds by combining the ideas of axioms of planes or spheres with the specific nature of these ambient spaces. For instance in Kaehlerian geometry K. Yano, I. Mogi, B. Y. Chen, K. Ogiue, K. Nomizu, S. I. Goldberg, E. M. Moskal, M. Harada, M. Kon, S. Yamagushi and M. Barros obtained characterizations of complex space forms in terms of axioms of holomorphic, antiholomorphic, coholomorphic, anti-invariant or *CR* planes or spheres. In particular, all submanifolds in these axioms are totally umbilical. The following Theorem gives a type of complex version of Theorem B which concerns nontotally umbilical submanifolds.

**Theorem E** (*B. Y. Chen and L. Verstraelen* [6]). *A Kaehlerian manifold of (real) dimension $m \geqslant 6$ with complex structure $J$ is a complex space form if and only if it satisfies the axiom of $J\xi$-quasiumbilical hypersurfaces, where $\xi$ is the hypersurface normal.*

Communicated by K. Yano, February 18, 1980.

S. Tachibana and T. Kashiwada proved that every geodesic hypersphere in a complex space form is $J\xi$-quasiumbilical [15]. The axiom of $J\xi$-quasiumbilical hypersurfaces was also studied by L. Vanhecke and T. J. Willmore, and the axiom of special $J\xi$-quasiumbilical hypersurfaces was studied first by S. Tashiro and S. Tachibana.

The main purpose of this article is to study axioms of submanifolds which are determined by conditions which are not exclusively extrinsic as in the case of the former axioms. More precisely, in §§3 and 4 we shall characterize the conformally flat spaces of dimension $> 4$ and the real space forms of dimension $> 3$ as the Riemannian manifolds which satisfy the *axiom of conformally flat quasiumbilical hypersurfaces* and of *Einsteinian hypercylinders*, respectively, that is, as the Riemannian manifolds $M$ for which there exists for each of their points $p$ and for every hyperplane section $H$ of their tangent space $T_pM$ at $p$ respectively a conformally flat quasiumbilical and an Einsteinian cylindrical hypersurface $N$ passing through $p$ and such that $T_pN = H$. It seems interesting to obtain further axioms of submanifolds which are determined by other extrinsic and intrinsic conditions or by purely intrinsic ones.

For a survey on axioms of submanifolds, see [16].

We thank Professor K. Yano for his kindly pointing out several facts about this theory and related topics.

## 2. Preliminaries

Let $M$ be a Riemannian manifold with metric tensor $\tilde{g}$, covariant differentiation $\tilde{\nabla}$ and curvature tensor $\tilde{R}$. Let $N$ be a submanifold of $M$ with induced metric tensor $g$, covariant differentiation $\nabla$ and curvature tensor $R$. The dimensions of $N$ and $M$ will be denoted by $n$ and $m$ respectively. Let $\eta$ be an arbitrary normal vector field and $X, Y, Z, U$ arbitrary tangent vector fields on $N$. Then the *formulas of Gauss* and *Weingarten* for $N$ in $M$ are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{1}$$

$$\tilde{\nabla}_X \eta = -A_\eta X + D_X \eta, \tag{2}$$

where $\sigma$, $A_\eta$ and $D$ are the second fundamental form, the second fundamental tensor corresponding to $\eta$ and the normal connection of $N$, respectively. One has the relation

$$\tilde{g}(\sigma(X, Y), \eta) = g(A_\eta X, Y). \tag{3}$$

The covariant derivative $\overline{\nabla}_X \sigma$ of $\sigma$ is defined by

$$\left(\overline{\nabla}_X \sigma\right)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{4}$$

and the *equations of Gauss* and *Codazzi* for the submanifold $N$ are given by [3]:

$$(5)\qquad R(X, Y; Z, U) = \tilde{R}(X, Y; Z, U) + \tilde{g}(\sigma(X, U), \sigma(Y, Z)) - \tilde{g}(\sigma(X, Z), \sigma(Y, U)),$$

$$(6)\qquad \tilde{R}(X, Y; Z, \eta) = \tilde{g}\big((\overline{\nabla}_X\sigma)(Y, Z), \eta\big) - \tilde{g}\big((\overline{\nabla}_Y\sigma)(X, Z), \eta\big).$$

In the following $N$ is assumed to be a hypersurface of $M$. Let $\xi$ be a unit hypersurface normal. Then the second fundamental form $\sigma$ can be written as

$$(7)\qquad \sigma(X, Y) = h(X, Y)\xi,$$

where $h$ is a scalar-valued symmetric 2-form on $N$, and relation (3) becomes

$$(8)\qquad h(X, Y) = g(A_\xi X, Y).$$

If on a hypersurface $N$ of dimension $n > 2$ there exist two functions $\alpha$, $\beta$ and a unit 1-form $\omega$ such that

$$(9)\qquad h = \alpha g + \beta\omega \otimes \omega,$$

that is, if $N$ has a principal curvature with multiplicity $n$ or $n - 1$, then $N$ is called a *quasiumbilical* hypersurface [7]. In particular, when $h$ is proportional to $g$ then $N$ is (totally) *umbilical*, and when $h$ vanishes identically then $N$ is *totally geodesic*. When $\beta$ is nonzero, then $N$ is called *W-quasiumbilical* whereby $W$ is the tangent vector field on $N$ for which $\omega(X) = g(W, X)$. In particular, when $\alpha$ is zero $N$ is called a cylindrical hypersurface or a *hypercylinder*.

Finally we recall the following characterizations of Riemannian manifolds which either have constant sectional curvature or are (locally) conformal to Euclidean space.

**Lemma F** (*E. Cartan* [2]). *A Riemannian manifold $M$ of dimension $> 2$ is a real space form if and only if $\tilde{R}(X, Y; Z, X) = 0$ for all orthonormal vector fields $X$, $Y$, $Z$ on $N$.*

**Lemma G** (*J. A. Schouten* [13]). *A Riemannian manifold $M$ of dimension $> 3$ is conformally flat if and only if $\tilde{R}(X, Y; Z, U) = 0$ for all orthonormal vector fields $X$, $Y$, $Z$, $U$ on $N$.*

## 3. Axiom of conformally flat quasiumbilical hypersurfaces

Using the conformal invariance of the notion of quasiumbilicity [4], it can be observed that for every point $p$ in any conformally flat space $M$ with $m > 3$ and for every $n(= m - 1)$-dimensional linear subspace $H$ of $T_pM$ there exist quasiumbilical hypersurfaces $N$ in $M$ such that $p \in N$ and $T_pN = H$. It follows from Theorem E that this property also holds for nonflat complex space forms. This implies that for $n = m - 1$ Theorem B can only partially be generalized from umbilical hypersurfaces to quasiumbilical ones, and that in order to obtain a property which is characteristic for conformally flat spaces it is necessary to impose an additional condition on the quasiumbilical hypersurfaces. In this respect we recall the following result.

**Theorem H** (*E. Cartan* [1], *J. A. Schouten* [12]). *A hypersurface $N$ of a conformally flat space $M$ of dimension $> 4$ is quasiumbilical if and only if it is conformally flat.*

Defining a Riemannian manifold $M$, $m > 3$, to satisfy the *axiom of conformally flat quasiumbilical hypersurfaces*, if for every point $p$ in $M$ and for every hyperplane section $H$ in $T_pM$ there exists a conformally flat quasiumbilical hypersurface $N$ passing through $p$ such that $T_pN = H$, we obtain the following.

**Theorem 1.** *A Riemannian manifold $M$ of dimension $m > 4$ is conformally flat if and only if it satisfies the axiom of conformally flat quasiumbilical hypersurfaces.*

*Proof.* First, from (5) and (9) we derive the equation of Gauss for any quasiumbilical hypersurface $N$:

$$\begin{aligned} R(X, Y; Z, U) = {} & R(X, Y; Z, U) \\ & + \alpha^2\{g(X, U)g(Y, Z) - g(X, Z)g(Y, U)\} \\ & + \alpha\beta\{g(X, U)\omega(Y)\omega(Z) + g(Y, Z)\omega(X)\omega(U) \\ & - g(Y, U)\omega(X)\omega(Z) - g(X, Z)\omega(Y)\omega(U)\}, \end{aligned} \tag{10}$$

where $X, Y, Z, U$ are arbitrary vector fields tangent to $N$. Now we assume that $M$ satisfies the axiom of conformally flat quasiumbilical hypersurfaces and that $\dim M = m > 4$. Then for any quadruple of orthonormal vectors $X, Y, Z, U$ at any point $p$ of $M$ there exists a conformally flat quasiumbilical hypersurface $N$ passing through $p$ such that $X, Y, Z, U$ belong to $T_pN$. Therefore from equation (10) of Gauss and Lemma G it follows that

$$\tilde{R}(X, Y; Z, U) = 0. \tag{11}$$

Again using Lemma G, (11) shows that $M$ is conformally flat.

In view of the comments preceding Theorem 1 the converse statement is trivial.

**Remarks.** 1. G. M. Lancaster proved that there exist conformally flat hypersurfaces in the Euclidean space $E^4$ which are not quasiumbilical [9].

2. A submanifold $N$ of dimension $n \geqslant 3$ and codimension $q$ in a Riemannian manifold $M$ is said to be *totally quasiumbilical* if with respect to $q$ mutually orthogonal normal directions $N$ has a principal curvature with multiplicity $\geqslant n - 1$. Concerning the relation between conformal flatness and

quasiumbilicity for submanifolds of codimension $> 1$ we mention the following results.

**Theorem I** (*B. Y. Chen and K. Yano* [7] [3]). *Every totally umbilical submanifold of dimension* $> 3$ *in a conformally flat space is conformally flat.*

**Theorem J** (*J. D. Moore and J. M. Morvan* [11]). *Every conformally flat submanifold of dimension* $n > 3$ *and codimension* $q \leqslant \min(4, n-3)$ *in a conformally flat space is totally quasiumbilical.*

**Theorem K** (*B. Y. Chen and L. Verstraelen* [5]). *Every conformally flat submanifold of dimension* $n > 3$ *and codimension* $q \leqslant n - 3$ *with flat normal connection in a conformally flat space is totally quasiumbilical.*

3. In a straightforward way Theorem 1 may be generalized as follows.

**Theorem 1′**. *A Riemannian manifold of dimension* $> 4$ *is conformally flat if and only if it satisfies the axiom of conformally flat totally quasiumbilical submanifolds of dimension* $> 3$.

## 4. Axiom of Einsteinian hypercylinders

The following intrinsic characterization of hypercylinders is an immediate consequence of (7) and equation (5) of Gauss [6].

**Lemma K**. *Let* $N$ *be a hypersurface in a Riemannian manifold* $M$. *Then* $N$ *is a hypercylinder if and only if the curvature tensors* $R$ *and* $\tilde{R}$ *of* $N$ *and* $M$ *satisfy* $R(X, Y; Z, U) = \tilde{R}(X, Y; Z, U)$ *for all vectors* $X, Y, Z, U$ *tangent to* $N$.

From Lemmas $F$, $G$ and $K$ it is clear that hypercylinders in respectively conformally flat spaces and real space forms are themselves respectively conformally flat spaces and real space forms. In some sense conversely, these lemmas also show that the conformally flat spaces and the real space forms can be characterized by an axiom of conformally flat hypercylinders and an axiom of hypercylinders with constant sectional curvature, respectively. Theorem 1 gives an improvement of the first one of these results. Theorem 2 does so for the second result. For its formulation we give the following definition: a Riemannian manifold $M$, $m > 3$, satisfies the *axiom of Einsteinian hypercylinders* if for every point $p$ in $M$ and every hyperplane section $H$ in $T_pM$ there exists an Einsteinian hypercylinder $N$ passing through $p$ such that $T_pN = H$.

**Theorem 2**. *A Riemannian manifold* $M$ *of dimension* $m > 3$ *is a real space form if and only if it satisfies the axiom of Einsteinian hypercylinders.*

*Proof*. Again it is sufficient to prove that if a Riemannian manifold satisfies the axiom of Einsteinian hypercylinders it is a real space form, the converse being obvious. To do so, and with the intention to use Lemma F, let

$X$, $Y$, $V$ be any triple of orthonormal vectors at any point of a Riemannian manifold $M$ of dimension $m > 3$ which satisfies this axiom. Then there exists an Einsteinian hypercylinder $N$ passing through $p$ and having $V$ as normal vector and $X$, $Y$ as tangent vectors. In particular Lemma K actually implies

$$(12) \qquad R(X, E_i; E_i, Y) = \tilde{R}(X, E_i; E_i, Y)$$

for any orthonormal basis $E_1 = X, E_2 = Y, E_3, \cdots, E_n$ of $T_pN$, $(i \in \{1, 2, \cdots, n\})$. Summation of (12) over $i$ yields

$$(13) \qquad S(X, Y) = \tilde{S}(X, Y) - \tilde{R}(X, V; V, Y),$$

where $S$ and $\tilde{S}$ denote the Ricci tensors of $N$ and $M$, respectively. Since $N$ is Einsteinian ($S = \lambda g$ for some constant $\lambda$) and $X$, $Y$ are perpendicular, we find that

$$(14) \qquad \tilde{S}(X, Y) = \tilde{R}(X, V; V, Y)$$

holds for all orthonormal vectors $X$, $Y$, $V$ at any point $p$ in $M$. Consequently if $\bar{V}$ is any other vector at $p$ such that $X$, $Y$, $\bar{V}$ are orthonormal, then

$$(15) \qquad \tilde{R}(X, \bar{V}; \bar{V}, Y) = \tilde{R}(X, V; V, Y).$$

Consider an orthonormal basis $F_1 = X, F_2 = Y, F_3 = V, F_4, \cdots, F_m$ of $T_pM$. Then (15) implies

$$\tilde{S}(X, Y) = (m - 2)\tilde{R}(X, V; V, Y).$$

From (14) and (16) it follows that

$$(17) \qquad (m - 3)\tilde{R}(X, V; V, Y) = 0,$$

which proves Theorem 2.

**Remarks.** 1. S. Kobayashi and K. Nomizu proved that a hypersurface in Euclidean space $E^m$, $m > 3$, is a hypercylinder if and only if it is Ricciflat [8]. Correspondingly in any Riemannian manifold $M$, $m > 3$, a hypersurface $N$ is a hypercylinder if and only if $III = II_H$ whereby $III$ and $II_H$ are the third fundamental form and the quadratic mean form of $N$ in $M$, respectively [17].

2. A submanifold $N$ of dimension $n > 3$ and codimension $q$ in a Riemannian manifold $M$ is said to be *totally cylindrical* if with respect to $q$ mutually orthogonal normal directions, 0 is a principal curvature of $N$ with multiplicity $\geqslant n - 1$. In a straightforward way Theorem 2 may be generalized as follows.

**Theorem 2′.** *A Riemannian manifold of dimension $> 3$ is a real space form if and only if it satisfies the axiom of Einsteinian totally cylindrical submanifolds of dimension $> 2$.*

3. We conclude this Section with the following particular case of Theorem E.

**Theorem 3.** *A Kaehlerian manifold $M$ of (real) dimension $m \geqslant 6$ with complex structure $J$ is flat if and only if it satisfies the axiom of $J\xi$-hypercylinders where $\xi$ is the hypercylinder normal.*

*Proof.* It is sufficient to show that if for each point $p$ in a Kaehlerian manifold $M$ with complex structure $J$ and of (real) dimension $\geqslant 6$ and for every hyperplane section $H$ in $T_pM$ with hyperplane normal section $\xi$ there exists a $J\xi$-hypercylinder $N$ passing through $p$ such that $T_pN = H$, then $M$ is flat. From Theorem E it is known that this condition forces $M$ to be a complex space form. Thus we need only to prove that its constant holomorphic sectional curvature actually vanishes. Therefore we derive from (6) and (9) the equation of Codazzi for any quasiumbilical hypersurface $N$:

$$\begin{aligned} R(X, Y; Z, \xi) &= (X\alpha)g(Y, Z) + (X\beta)\omega(Y)\omega(Z) + \beta(\nabla_X\omega)(Y)\omega(Z) \\ &+ \beta\omega(Y)(\nabla_X\omega)(Z) - (Y\alpha)g(X, Z) - (Y\beta)\omega(X)\omega(Z) \\ &- \beta(\nabla_Y\omega)(X)\omega(Z) - \beta\omega(X)(\nabla_Y\omega)(Z). \end{aligned} \tag{18}$$

In the present situation $\alpha$ vanishes, and $\omega$ is the 1-form which is dual with respect to $J\xi$, that is,

$$\omega(Z) = g(J\xi, Z) \tag{19}$$

for all $Z$ tangent to the $J\xi$-hypercylinder $N$. Since for all $X$, $Y$ tangent to $N$

$$Y\omega(X) = (\nabla_Y\omega)(X) + \omega(\nabla_Y X), \tag{20}$$

it follows from (19) that

$$(\nabla_Y\omega)(X) = g(A_\xi Y, JX). \tag{21}$$

Let $U$ be any vector field tangent to $N$ such that $\omega(U) = 0$. Then $A_\xi U = 0$, and (21) becomes

$$(\nabla_U\omega)(Y) = 0. \tag{22}$$

Making use of (22) in equation (18) of Codazzi where $\alpha = 0$, we find that

$$\tilde{R}(U, JU; J\xi, \xi) = 0, \tag{23}$$

which ends the proof.

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